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Birzeit University
Mathematics Department
Math331
First Exam

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Name:...
Section: (1)

First Semester 2019/2020
Number:..............
Date: 16/10/2019

Question I [10 points]. **Tchebycheff polynomials of the second kind** are defined by

$$U_n(x) = \frac{\sin\left[(n+1)\cos^{-1} x\right]}{\sqrt{1-x^2}}, \ n \in \mathbb{N}.$$

Show that the set $\{U_n(x)\}$, $-1 \le x \le 1$, is orthogonal relative to the weight function $\omega(x) = \sqrt{1-x^2}$.

We need to show that $\langle U_n(x), U_m(x)\rangle = 0, \ \forall m \neq n$.

Indeed, $\langle U_n(x), U_m(x)\rangle = \int_{-1}^{1} U_n(x)\, U_m(x)\, \omega(x)\, dx$

2

$$= \int_{-1}^{1} \frac{\sin\left[(n+1)\cos^{-1}x\right]}{\sqrt{1-x^2}} \cdot \frac{\sin\left[(m+1)\cos^{-1}x\right]}{\cancel{\sqrt{1-x^2}}} \cancel{\sqrt{1-x^2}}\, dx$$

Let $y = \cos^{-1}x \Rightarrow dy = \frac{-1}{\sqrt{1-x^2}}\, dx$ 2

$x = -1 \Rightarrow y = \pi$, $x = 1 \Rightarrow y = 0$

$$= \int_0^{\pi} \sin(n+1)y\, \sin(m+1)y\, dy$$ 2

$$= \frac{1}{2}\int_0^{\pi} \left[\cos(n-m)y - \cos(n+m+2)\cdot y\right] dy$$ 2

$$= \frac{1}{2}\left[\frac{\sin(n-m)y}{n-m} - \frac{\sin(n+m+2)y}{n+m+2}\right]_0^{\pi}$$ 2

$= 0, \ \forall m \neq n.$

Question II [**14 points**]. Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \leq x < 1. \end{cases}$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

3

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_0^1 x\cos(n\pi x)\,dx$$

3

$$= \left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x)\right]_0^1$$

$$= \frac{(-1)^n - 1}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_0^1 x\sin(n\pi x)\,dx$$

3

$$= \left[\frac{-x}{n\pi}\cos(n\pi x) + \frac{1}{n^2\pi^2}\sin(n\pi x)\right]_0^1$$

$$= \frac{-(-1)^n}{n\pi}$$

therefore, the Fourier series of f is

5

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos(n\pi x) - \frac{(-1)^n}{n\pi}\sin(n\pi x)\right]$$

Question III [8+4 points].

(a) Find the **half-range Fourier cosine** expansion of $f(x) = \sin 4x, \quad 0 \le x < \frac{\pi}{8}$.

(b) Use **part (a)** to show that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}.$$

(a) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(8nx)$$ 1

$$a_0 = \frac{16}{\pi}\int_0^{\frac{\pi}{8}} \sin 4x \, dx = -\frac{16}{\pi} \left.\frac{\cos 4x}{4}\right|_0^{\pi/8} = \frac{4}{\pi}$$ 2

$$a_n = \frac{16}{\pi}\int_0^{\pi/8} \sin 4x \cos(8nx)\, dx$$

$$= \frac{8}{\pi}\int_0^{\pi/8} \left[\sin(4+8n)x + \sin(4-8n)x\right] dx$$

$$= \frac{-8}{\pi}\left[\frac{\cos(4+8n)x}{4+8n} + \frac{\cos(4-8n)x}{4-8n}\right]_0^{\pi/8}$$

$$= \frac{8}{\pi}\left[\frac{1}{4+8n} + \frac{1}{4-8n}\right] = \frac{4}{\pi}\frac{1}{1-4n^2}$$ 4

$$\therefore f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos(8nx)$$ 1

(b) put $x = 0$ 2

$$0 = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$$ 2

Question IV [15+3+4+2 points]. Consider the following eigenvalue problem

$$\begin{cases} y'' - 4y' + \lambda y = 0, & 0 < x < 1 \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

(a) Find the eigenvalues and associated eigenfunctions.

(b) Put the differential equation in **self-adjoint** form and find its **weight function**.

(c) Use **(a)** and **(b)** above to obtain the eigenfunctions expansion of $f(x) = e^{2x}$, $0 < x < 1$.

(d) Use **(c)** above to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots = \frac{\pi}{4}.$$

(a) The auxiliary equation is $m^2 - 4m + \lambda = 0$

1 $\Rightarrow m = 2 \pm \sqrt{4-\lambda}$

2 <u>Case I.</u> $4 - \lambda = 0 \Rightarrow m = 2, 2 \Rightarrow y = c_1 e^{2x} + c_2 x e^{2x}$

1 • $y(0) = 0 \Rightarrow c_1 = 0 \Rightarrow y = c_2 x e^{2x}$

1 • $y(1) = 0 \Rightarrow c_2 e^2 = 0 \Rightarrow c_2 = 0 \Rightarrow y \equiv 0$ trivial.

<u>Case II.</u> $4 - \lambda > 0$, so we put $4 - \lambda = \alpha^2$ $(\alpha > 0)$

$\Rightarrow m = 2 \pm \alpha = 2 + \alpha$ or $2 - \alpha$

2

$\Rightarrow y = c_1 e^{(2+\alpha)x} + c_2 e^{(2-\alpha)x}$

1 • $y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow \boxed{c_2 = -c_1}$

$\Rightarrow y(x) = c_1 e^{(2+\alpha)x} - c_1 e^{(2-\alpha)x}$

• $y(1)=0 \Rightarrow c_1 e^{2+\alpha} - c_1 e^{2-\alpha} = 0$

1

$$\Rightarrow c_1 \underbrace{\left(e^{2+\alpha} - e^{2-\alpha}\right)}_{\neq 0} = 0 \Rightarrow c_1 = 0$$

$$\Rightarrow c_2 = 0$$

$$\Rightarrow y(x) \equiv 0 \quad \text{trivial solution.}$$

Case III. $4-\lambda<0$, so we put $4-\lambda=-\alpha^2$, $\alpha>0$

1

$$\Rightarrow m = 2 \pm \sqrt{-\alpha^2} = 2 \pm \alpha i$$

$$\Rightarrow y(x) = e^{2x}\left[c_1 \cos\alpha x + c_2 \sin\alpha x\right]$$

1 • $y(0)=0 \Rightarrow c_1=0 \Rightarrow \boxed{y(x) = c_2 e^{2x} \sin\alpha x}$

• $y(1)=0 \Rightarrow c_2 e^2 \sin\alpha = 0$

Since we aim to get nontrivial solutions and $\sin\alpha$ may be zero, we consider $c_2 \neq 0$ and

2 $\sin\alpha = 0 \Rightarrow \alpha_n = n\pi$, $n=1,2,3,\ldots$

$\Rightarrow$ the nontrivial solutions are $y = c_2 e^{2x} \sin(n\pi x)$

So, as $4-\lambda=-\alpha^2$, this BVP has eigenvalues

1 $\lambda_n = 4 + n^2\pi^2$, $n=1,2,3,\ldots$

and eigenfunctions: $\phi_n(x) = e^{2x}\sin(n\pi x)$, $n = 1, 2, 3, \ldots$

1

(b) Multiply both sides of the eq. by $I.F = e^{\int -4dx} = e^{-4x}$:

1

$$\underbrace{e^{-4x}y'' - 4e^{-4x}y'} + \lambda e^{-4x}y = 0$$

$$\frac{d}{dx}\left(e^{-4x}y'\right) + \left(0 + e^{-4x}\lambda\right)y = 0$$ 1

weight function $= \boxed{w(x) = e^{-4x}}$ 1

(c) $f(x) = e^{2x} = \sum_{n=1}^{\infty} a_n \phi_n(x)$, where 1

$\phi_n(x) = e^{2x}\sin(n\pi x)$ and

$$a_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \frac{\int_0^1 e^{2x}e^{2x}\sin(n\pi x)\, e^{-4x}dx}{\int_0^1 \left[e^{2x}\sin(n\pi x)\right]^2 e^{-4x}dx}$$

2

$$= \frac{\int_0^1 \sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx} = \frac{\left[1-(-1)^n\right]/n\pi}{1/2}$$

$$= \frac{2\left(1-(-1)^n\right)}{n\pi}$$

1

$$\therefore \boxed{e^{2x} = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} e^{2x} \sin(n\pi x)}, \quad 0 < x < 1$$

1

(d) put $x = 1/2$ in (c),

$$\cancel{e} = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \cancel{e} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{2}{1}\sin\frac{\pi}{2} + 0 + \frac{2}{3}\sin\left(\frac{3\pi}{2}\right) + 0 + \frac{2}{5}\sin\left(\frac{5\pi}{2}\right) + \cdots$$

$$\frac{\pi}{2} = 2(1) + \frac{2}{3}(-1) + \frac{2}{5}(1) - \cdots$$

Divide by 2 :

1

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Question V [5+5 points]. If the linear operator L is defined by

$$Ly = a(x)y''(x) + b(x)y'(x) + c(x)y(x)$$

and the linear operator L^* defined so that

$$L^*y = [a(x)y(x)]'' - [b(x)y(x)]' + c(x)y(x)$$

then we say that L^* is the **adjoint** of L. If $L = L^*$, then the operator L is called **self-adjoint**.

(a) Show that if

$$Ly = p(x)y''(x) + p'(x)y'(x) + [q(x) + \lambda r(x)]y(x)$$

then L is **self-adjoint**.

(b) For L defined in **(a)** show that

$$zLy - yLz = [p(zy' - z'y)]'$$

is satisfied if all derivatives exists.

2

(a) $L^*y = [p(x)y(x)]'' - [p'(x)y(x)]' + [q(x) + \lambda r(x)]y(x)$

$= [p'(x)y(x) + p(x)y'(x)]' - p''(x)y(x) - p'(x)y'(x)$ 2

$+ [q(x) + \lambda r(x)]y(x)$

$= \cancel{p''(x)y(x)} + p'(x)y'(x) + \cancel{p'(x)y'(x)} + p(x)y''(x)$

1 $- \cancel{p''(x)y(x)} - \cancel{p'(x)y'(x)} + [q(x) + \lambda r(x)]y(x)$

$= p(x)y''(x) + p'(x)y'(x) + [q(x) + \lambda r(x)]y(x)$

$= Ly.$

(b) $[p(zy' - z'y)]'$

$= p'(zy' - z'y) + p(\cancel{z'y'} + zy'' - \cancel{z'y'} - z''y)$ 2

$= z(py'' + p'y') - y(pz'' + p'z')$

$= z(py'' + p'y' + [q(x) + \lambda r(x)]y)$ 2

$- y(pz'' + p'z' + [q(x) + \lambda r(x)]z)$

1 $= zLy - yLz$

Good Luck

وفقكم الله